and fills the region $\Omega(C^*) \setminus \Omega(C_5)$ for $C^* \geqslant C_5$.

The trajectory, leaving an arbitrary point (x_0, y_0), after a finite segment of time enters into the band $-b \leqslant y \leqslant b$, i.e. it turns out to be in some region $\Omega(C)$. In fact, for the sake of definiteness let $y_0 > b$. Since $G(x) + \alpha_0 x \to \infty$ for $x \to \infty$, the curve $\Gamma(^1/_2 y_0^2 + G(x_0) + \alpha_0 x_0;\ \alpha_0)$ for $x > x_0$ intersects the straight line $y = b$ Then by virtue of (3) and the inequality $b > E_{20}$ the trajectory also intersects this line and enters into the band $-b \leqslant y \leqslant b$.

From inequalities (8), (10), (11) and $|E_2(t)| \leqslant E_{20}$ it follows that the trajectories cross the curves $\omega(C)$ from the outside into the region $\Omega(C)$ for any $C \geqslant C_5$. Consequently, in the region $\Omega(C^*) \setminus \Omega(C_5)$ $(C^* \geqslant C_5)$ the quantity C decreases monotonically along the trajectory. If the existence of the limit $C^+ \geqslant C_5$ is assumed here, then this will indicate that the trajectory winds up from the outside onto the curve $\omega(C^+)$. In particular, in the region $x \geqslant a$, $-E_{20} > y > -b$ the function $x^{\cdot}|_{(2)}$ becomes arbitrarily close to zero. Howevet, this is in contradiction to the first equation of system (2) and the inequality* $|E_2(t)| \leqslant E_{20}$.

Thus, in the course of time all trajectories of system (2) get into the region $\Omega(C_5)$ and subsequently remain in it. This completes the proof of the theorem.

Note: Under the conditions of the theorem the requirement $|g(x)| \leqslant g_0 < \infty$ is essential. Thus, for the equation

$$x^{\cdot\cdot} + \operatorname{sign} x^{\cdot} + x = \sin t$$

the statement of the theorem is not valid [3].

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BIFURCATIONS IN THE VICINITY OF A "FUSED FOCUS"

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Conditions are presented for the existence of bifurcation of a singular point of the type of a "fused focus". The fusing is accomplished with ordinary trajectories under the assumption that the general integrals are known for both systems forming the "fused system".

In the approximation of analytical characteristics in the equations of motion of dynamic systems by piecewise linear or relay functions on the lines of fusing, singular points can arise which are fused from ordinary or singular trajectories of systems to be fused. When the parameters of the system change, analogies arise here to bifurcations which are known for systems with analytical right-hand sides (birth of limit cycles from the fused focus, from the fused loop of the separatrix etc.).

Let the line of fusing be $x = 0$ and let the following system be defined in the xy - plane

$$\frac{dx}{dt} = P_i(x, y), \quad \frac{dy}{dt} = Q_i(x, y) \quad (i = 1,2) \tag{1}$$

$$i = 1 \text{ for } x < 0, \quad i = 2 \text{ for } x > 0$$

It is not difficult to show that the location of the trajectories in the vicinity of the origin of coordinates is as in Fig. 1 ($Q_1(0, 0) > 0$) or Fig. 2 ($Q_1(0, 0) < 0$), and that the origin of coordinates is a singular point of the type of a fused focus if the fulfillment of the following conditions is required

$$P_1(0, 0) = P_2(0, 0) = 0$$

$$\frac{P_{1y}'(0, 0)}{Q_1(0, 0)} > 0, \quad \frac{P_{2y}'(0, 0)}{Q_2(0, 0)} < 0, \quad Q_1(0, 0)\, Q_2(0, 0) < 0 \tag{2}$$

Conditions for which the fused focus is stable or unstable will be obtained later. Let

$$F_1(x, y) = C_1 \quad (x < 0), \qquad F_2(x, y) = C_2 \qquad (x > 0)$$

be the general integrals of the system to be fused on the straight line $x = 0$. Point transformation of the half-line $y > 0$ into itself, carried out on trajectories of system (1) (Fig. 3), is given in a parametric form (u is the parameter) by correspondence functions

$$F_1(0, v_1) = F_1(0, u), \qquad F_2(0, v_2) = F_2(0, u) \tag{3}$$

Let us examine some properties of functions given by the following equation:

$$F(v) = F(u) \tag{4}$$

1. The graph of the function defined by Eq. (4) is symmetric with respect to the bisectrix $v = u$ and coincides with it if the function is monotonic.

2. If $u = u_0$ is the extremum point of the function $F(u)$, then the point (u_0, u_0) is a double singular point of the function defined by Eq. (4). The graph of this function in the vicinity of the point (u_0, u_0) consists of two branches, one of which coincides with the bisectrix $v = u$, the other represents a decreasing function $v = \varphi(u)$, which is symmetric with respect to the bisectrix $v = u$.

These two properties are geometrically obvious and easy to prove.

3. If the function $F(u)$ in the point $u = u_0$ has derivatives to the order including $(n + 1)$, and if $F'(u_0) = 0$ and $F''(u_0) \neq 0$, then the function $v = \varphi(u)$ in the point $u = u_0$ has derivatives to the order including n. These derivatives can be computed through the usual application of the rules of analysis. Expressions are presented here for the first four derivatives

$$\varphi'(u_0) = -1, \qquad \varphi''(u_0) = -\frac{2}{3}\frac{F'''(u_0)}{F''(u_0)}, \qquad \varphi'''(u_0) = -\frac{3}{2}[\varphi''(u_0)]^2$$

$$\varphi^{(4)}(u_0) = \frac{3}{5}\frac{F^{(5)}(u_0)}{F'''(u_0)}\varphi''(u_0) + 3\frac{F^{(4)}(u_0)}{F'''(u_0)}[\varphi''(u_0)]^2 + 6[\varphi''(u_0)]^3 \tag{5}$$

Note. It can be shown that derivatives of uneven order of the function $v = \varphi(u)$ are expressed through derivatives of this function of preceding orders. Therefore the first nonzero derivative must be of even order. For proof it is sufficient to turn the axes by the angle $\alpha = \pi/4$ and take advantage of the fact that uneven order derivatives of an even function become zero at the origin of coordinates.

4. If two functions $v = \varphi_1(u)$ and $v = \varphi_2(u)$, which are defined by equations $F_1(v) = F_1(u)$ and $F_2(v) = F_2(u)$ satisfy the conditions formulated under property 3, then for their difference $z(u)$ the following statements are valid:

a) The function $z(u)$ in the point $u = u_0$ has a zero of even multiplicity (this follows from the Note to property 3).

b) The function $z(u)$ has an uneven number of zeros, and the number of zeros less than u_0 is equal to the number of zeros greater than u_0. This follows from the symmetry of curves $v = \varphi_1(u)$ and $v = \varphi_2(u)$ with respect to the straight line $v = u$.

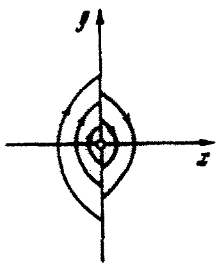

Fig. 1.

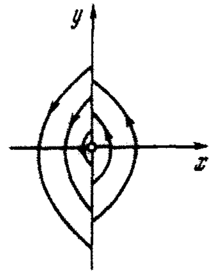

Fig. 2.

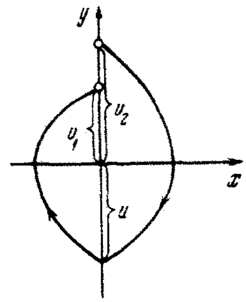

Fig. 3.

Let us now return to correspondence functions (3). If the additional assumption is made that $F_i''(0, 0) \neq 0$ $(i = 1, 2)$, then all properties enumerated above are true for correspondence functions. In this connection $u_0 = 0$, and graphs of correspondence curves $v = \varphi_1(u)$ and $v = \varphi_2(u)$, which are defined by Eqs. (3), are the parts of curves located above the straight line $v = u$, because in Fig. 3 it is assumed that $u \dot{<} 0$.

Let us examine the difference of correspondence functions

$$z(u) = \varphi_1(u) - \varphi_2(u)$$

The zeros of functions $z(u)$, excluding the point $u = 0$, correspond to limit cycles of the fused system while the point $u = 0$ corresponds to the singular point of the type of a fused focus. In this connection, apparently, it is necessary to examine the zeros only on the positive or only on the negative half-axes u (in our case according to Fig. 3 only on the negative half-axis u). The expansion of function $z(u)$ in powers of u has the following form by virtue of statement (a) of property 4:

$$z(u) = \alpha_{2k} u^{2k} + \ldots . \tag{6}$$

The sign of the coefficient α_{2k} together with the sign of the quantity $Q_1(0, 0)$ determines the stability or the instability of the fused focus. If α_{2k} and $Q_1(0, 0)$ have different signs, the fused focus is stable. If the signs are the same, the fused focus is unstable. Coefficients in the expansion (6) play the role of Liapunov focal quantities for a fused system and determine the behavior of the trajectory of the fused system in the vicinity of the singular point. Let us assume that coefficients of the expansion (6) and their derivatives are continuously dependent on parameter λ. For some value of parameter $\lambda = \lambda_0$ let the condition $\alpha_2(\lambda_0) \neq 0$ be valid. Then there exists such an interval of variation of parameter $\hat{\lambda}$, containing the point $\lambda = \lambda_0$, that for all values of λ from this interval the inequality $\alpha_2(\lambda_0) \geqslant r^2 > 0$ is satisfied. But in this case a neighborhood of the point $u = 0$ exists such that in this neighborhood the function $z(u)$ retains its sign for all values of λ from the above mentioned interval. In analogy to usual definitions we say that the singular point of a fused system is a singular point of the simple fused focus type.

Let now $Q_1(0, 0) > 0$, $\alpha_2(\lambda_0) = 0$ (and consequently $\alpha_3(\lambda_0) = 0$), and $\alpha_4(\lambda_0) > 0$. We assume for the sake of definiteness $\alpha_2'(\lambda_0) < 0$. Then the following statements are valid:

1. It is possible to find such values $\varepsilon > 0$ and $\lambda_1 < \lambda_0$ that for all u from the interval $(-\varepsilon, 0)$ and for all λ from the interval (λ_1, λ_0) the function $z(u)$ is positive.

2. It is possible to find such values $\varepsilon > 0$ and $\lambda_2 > \lambda_0$ that for all λ from the interval (λ_0, λ_2) the function $z(u)$ in the interval $(-\varepsilon, 0)$ has only one zero $u = u_1$ ($z(u) > 0$ in the interval $(-\varepsilon, u_1)$, $z(u) < 0$ in the interval $(u_1, 0)$).

It follows from these statements that on the phase plane xy of the fused system the following changes take place, when the parameter λ increases: For values $\lambda < \lambda_0$ and sufficiently close to λ_0 the point $(0, 0)$ is a singular point of the type of a simple unstable fused focus. For $\lambda = \lambda_0$ the point $(0,0)$ becomes a composite fused focus (unstable). For $\lambda > \lambda_0$ the composite fused focus becomes a simple stable fused focus surrounded by an unstable limit cycle. For the opposite change of parameter the limit cycle contracts towards the singular point, while the focus becomes unstable again. For other assumptions with respect to signs of $\alpha_4(\lambda_0)$ and $\alpha_2'(\lambda_0)$ the following cases are possible:

$$Q_1(0, 0) > 0, \quad \alpha_2(\lambda_0) = 0, \quad \alpha_4(\lambda_0) > 0, \quad \alpha_2'(\lambda_0) > 0$$

As λ increases, the unstable limit cycle contracts towards the stable fused focus.

$$Q_1(0, 0) > 0, \quad \alpha_2(\lambda_0) = 0, \quad \alpha_4(\lambda_0) < 0, \quad \alpha_2'(\lambda_0) > 0$$

As λ increases, a stable limit cycle is generated from the stable composite focus

$$Q_1(0, 0) > 0, \quad \alpha_2(\lambda_0) = 0, \quad \alpha_4(\lambda_0) < 0, \quad \alpha_2'(\lambda_0) < 0$$

As λ increases, the stable limit cycle contracts towards the unstable fused focus.

These conclusions are analogous to those which were obtained for dynamic systems with analytical right sides [1].

In the general case

$$\alpha_2(\lambda_0) = \alpha_4(\lambda_0) = \ldots \alpha_{2k}(\lambda_0) = 0, \quad \alpha_{2k+2}(\lambda_0) \neq 0$$

with appropriate introduction of the parameter, k and not more than k limit cycles can generate.

Example. Let us examine the equation [2]

$$\frac{d\rho}{d\varphi} = \frac{2\rho(\lambda - \mu\rho - \sin\varphi)}{\rho - \cos\varphi} \tag{7}$$

with the following approximations:

$$\sin\varphi \sim \begin{cases} -1 \\ +1 \end{cases} \qquad \cos\varphi \sim \begin{cases} 1 + \varphi & \text{for} \quad -\pi < \varphi < 0 \\ 1 - \varphi & \text{for} \quad 0 < \varphi < \pi \end{cases}$$

Changing to variables $y = \rho - 1$, $x = \varphi$, we obtain the fused system

$$\frac{dx}{dt} = y - x = P_1, \qquad \frac{dy}{dt} = 2(y + 1)(\lambda - \mu + 1 - \mu y) = Q_1 \qquad (-\pi < x < 0)$$

$$\frac{dx}{dt} = y + x = P_2, \qquad \frac{dy}{dt} = 2(y + 1)(\lambda - \mu - 1 - \mu y) = Q_2 \qquad (0 < x < \pi) \tag{8}$$

Conditions (2) are satisfied if

$$Q_1(0,0) = 2(\lambda - \mu + 1) > 0, \qquad Q_2(0,0) = 2(\lambda - \mu - 1) < 0 \tag{9}$$

Under these conditions the point (0,0) is a singular point of the fused focus type with motion of the representing point along the trajectory in the clockwise direction because $Q_1(0, 0) > 0$. The general integrals of system (8) are

$$F_i(x, y) = 2x\left[\frac{\pm(a-\mu y)}{(y+1)}\right]^{\mp K} \mp \int\limits_{y_0}^{y} t(t+1)^{\pm K-1}[\pm(a-\mu t)]^{\mp K-1}\,dt = C_i$$

Here the upper signs correspond to $x < 0$, the lower signs to $x > 0$. The following notation is used:

$$a = \lambda - \mu \pm 1, \qquad k = \frac{1}{2(\lambda \pm 1)}$$

From Eqs. (5) we obtain

$$\varphi_1''(0) = -\frac{4}{3}\,\frac{4\mu - 2\lambda - 1}{2(\lambda - \mu + 1)}, \qquad \varphi_2''(0) = -\frac{4}{3}\,\frac{-4\mu + 2\lambda - 1}{2(-\lambda + \mu + 1)}$$

The first term in the expansion (6) will be

$$\alpha_2 = \frac{1}{2!}\{\varphi_1''(0) - \varphi_2''(0)\} = \frac{8}{3}\,\frac{(3\mu - \lambda)}{Q_1(0,0)\,Q_2(0,0)}$$

Since $Q_1(0, 0) > 0$ (9), the point (0, 0) is a stable focus for $\lambda < 3\mu$ ($\alpha_2 < 0$) and an unstable focus for $\lambda > 3\mu$. For $\lambda = 3\mu$ the point (0, 0) is a composite fused focus and following α_2 in the expansion (6), the coefficient α_3 becomes zero. Using Eqs. (5), the calculation of α_4 gives

$$\alpha_4 = \frac{1}{4!}\{\varphi_1^{(4)}(0) - \varphi_2^{(4)}(0)\} = \frac{\mu}{5}\,\frac{(1 + 4\mu^2)}{(1 - 4\mu^2)^2} > 0$$

It follows from this that when the stability of the singular point of system (8) changes, not more than one limit cycle can appear. When the parameter μ grows beyond the bifurcation value, which is determined by the condition $\lambda = 3\mu$, one unstable limit cycle appears from the composite fused focus.

A comparison of results of the investigation of bifurcation in the vicinity of the singular point using the approximations (system (8)) and the initial system (7) (see [3]), shows that in this case the character of the possible bifurcations does not change in the vicinity of the singular point of the type of a focus. For other bifurcations in Eq. (7), which are connected with the appearance or disappearance of limit cycles (from condensation of trajectories, or from the loop of the separatrix) the utilization of approximations opens the possibility to obtain analytical conditions expressed through the parameters of the system. These conditions cannot be obtained for the initial analytical system.

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ON THE THEORY OF TWO-WAY TRAFFIC FLOW

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The propagation of a shock wave, the process of decay of the wave, and the motion of a type of simple wave in a two-way uniform symmetric traffic flow are studied on the bases of the hydrodynamic model postulated by J. Blick and G. Newell [1]. An expression is obtained that connects the flow parameters at the front of the shock wave. The decay of the wave is investigated in the vicinity of the head part by using expressions of the parameters in power series in a small quantity. Terms of the expansions are calculated that characterize the rate of change of the wave profile and its curvature at the wave front.

1. Formulation of the problem. In the hydrodynamic theory of two-way uniform symmetric traffic flow two continuity equations

$$\frac{\partial p}{\partial t}+\frac{\partial (pu)}{\partial x}=0, \qquad \frac{\partial q}{\partial t}+\frac{\partial (qv)}{\partial x}=0 \tag{1.1}$$

are used to determine the form of dependence on the independent variables of the average speeds u and v and densities p and q of two homogeneous streams of traffic moving in opposite directions. The system of equations (1.1) is completed by the two empirical relations

$$u=v_0-\alpha p-\beta q, \qquad v=u_0+\beta p+\alpha q \tag{1.2}$$

which represent the average speeds as functions of the densities of both streams.

The region of physically acceptable solutions of the system (1.1), (1.2) is bounded; it can be represented as the union of the regions of hyperbolicity and ellipticity of the system of equations [1, 2]. The laws of motion and growth of initially small perturbations of the flow parameters and the magnitude of the time interval required for the transformation of a weak discontinuity into a shock wave are obtained in [3].

On the generated shock wave the following conditions are satisfied [1] :

$$\begin{gathered} p_0\,[u\,(p_0,\,q_0)-w]=p\,[u\,(p,\,q)-w] \\ q_0\,[v\,(p_0,\,q_0)-w]=q\,[v\,(p,\,q)-w] \end{gathered} \tag{1.3}$$

Here the initial unperturbed state is denoted by subscript zero and w, denotes the speed of the shock wave. Eliminating w we obtain the equation of the shock polar

$$\frac{pu\,(p,\,q)-p_0u\,(p_0,\,q_0)}{p-p_0}=\frac{qv\,(p,\,q)-q_0v\,(p_0,\,q_0)}{q-q_0} \tag{1.4}$$

2. Motion of the shock wave. In equations (1.1), (1.2) we introduce dimensionless quantities according to the equations [1]